

Coherent destruction of tunneling, dynamic localization and the Landau-Zener formula

Yosuke Kayanuma¹ and Keiji Saito²

¹*Department of Mathematical Sciences, Graduate School of Engineering
Osaka Prefecture University, Sakai 599-8531, Japan*

²*Department of Physics, Graduate School of Science
University of Tokyo, Bunkyo-ku, Tokyo 113-0033, Japan*

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We clarify the internal relationship between the coherent destruction of tunneling (CDT) for a two-state model and the dynamic localization (DL) for a one-dimensional tight-binding model, under the periodical driving field. The time-evolution of the tight-binding model is reproduced from that of the two-state model by a mapping of equation of motion onto a set of SU(2) operators. It is shown that DL is effectively an infinitely large dimensional representation of the CDT in the SU(2) operators. We also show that both of the CDT and the DL can be interpreted as a result of destructive interference in repeated Landau-Zener level-crossings.

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The coherent control of quantum dynamics of electrons by a periodically oscillating external field has been one of the subjects of considerable interest both in nanoscale solid state physics[1], and in molecular physics under laser fields[2]. The interest is now extended to the trapped atoms in Bose-Einstein condensates[3], the localized spins in molecular magnets[4], the Cooper pairs in Josephson qubits[5], to name only a few. It should be noted that, even when the static properties of a quantum system is well known, its response to an explicitly time-dependent driving field may be nontrivial and, in some cases, poses a quite interesting problem. The phenomena known as the coherent destruction of tunneling (CDT) [6] and dynamic localization (DL) [7] are such typical nontrivial phenomena. Note that the CDT was originally found by Grossmann et al.[6] for a model of double-well potential, but it has been made clear that the essential mechanism of the phenomenon can be well understood by a two-state model which represents the quantum dynamics between the lowest two states localized to each well [8, 9]

In both CDT and DL, the initial localized quantum state never diffuses under a periodic external field. In this aspect, these phenomena are similar. However, there are also some dissimilarities. The DL is an exact result obtained in an infinite driven system and is valid irrespective of the magnitudes of the transfer matrix element. On the other hand, the CDT is derived approximately in an extreme case of a small value of the transfer matrix element. In the CDT, the initial distribution is frozen, but in the DL, the distribution oscillates around the initial value. Thus the relation between the CDT and the DL has been controversial[10, 11].

In this Rapid Communication, we study the relationship between these remarkable phenomena in a unified way. One may study this problem by assuming a tight-binding model with a finite length, and by observing the change of the behavior of the electron according to the change of the chain length[11]. However, the equation of

motion for a finite linear chain model does not allow for the analytical solution, so that the analysis inevitably becomes a numerical one. We present here a new approach to this problem, which shed light upon the internal relationship between the CDT and the DL. It will be shown that the DL is an infinitely large dimensional representation of a generalized version of the CDT, and in fact they are closely related each other.

Let the two-state system $|1\rangle$ and $|2\rangle$ be under an external field and driven by the Hamiltonian,

$$H_1(t) = \frac{E(t)}{2} (|1\rangle\langle 1| - |2\rangle\langle 2|) + \gamma (|1\rangle\langle 2| + |2\rangle\langle 1|), \quad (1)$$

where γ is a constant tunneling matrix element. The Schrödinger equation ($\hbar = 1$), $i d/dt |\psi(t)\rangle = H_1(t) |\psi(t)\rangle$, is cast into the form

$$\begin{aligned} i \frac{d}{dt} a_1(t) &= \frac{E(t)}{2} a_1(t) + \gamma a_2(t), \\ i \frac{d}{dt} a_2(t) &= -\frac{E(t)}{2} a_2(t) + \gamma a_1(t), \end{aligned} \quad (2)$$

in the representation $|\psi(t)\rangle = a_1(t)|1\rangle + a_2(t)|2\rangle$. Although this is the simplest equation of quantum dynamics, it cannot be solved analytically for general functional forms of $E(t)$. Grossmann and Hänggi [8] and Llorente and Plata [9] pointed out that for a sinusoidal time-dependence of the driving field $E(t) = E_0 \cos(\omega t)$, Eq.(2) is solved approximately in the limit of rapid modulation $\omega \gg \gamma$. By substituting $a_1(t) = \exp[-i(E_0/2\omega) \sin(\omega t)] c_1(t)$, $a_2(t) = \exp[i(E_0/2\omega) \sin(\omega t)] c_2(t)$, Eq.(2) is rewritten as

$$\begin{aligned} i \frac{d}{dt} c_1(t) &= \gamma \exp[i(E_0/\omega) \sin(\omega t)] c_2(t), \\ i \frac{d}{dt} c_2(t) &= \gamma \exp[-i(E_0/\omega) \sin(\omega t)] c_1(t). \end{aligned} \quad (3)$$

In the limit $\omega \gg \gamma$, the above equation is integrated approximately for a short period $2\pi/\omega$ by assuming that

$c_1(t)$ and $c_2(t)$ are constant, since the rapidly oscillating terms are separated out as the phase factors. This is the inverse adiabatic approximation. We obtain

$$\begin{aligned} i \frac{d}{d\tau} c_1(\tau) &= \gamma J_0(E_0/\omega) c_2(\tau), \\ i \frac{d}{d\tau} c_2(\tau) &= \gamma J_0(E_0/\omega) c_1(\tau), \end{aligned} \quad (4)$$

where τ is a coarse-grained time by the unit of $2\pi/\omega$, and $J_0(E_0/\omega)$ is the zeroth order Bessel function:

$$J_0(E_0/\omega) \equiv \frac{\omega}{2\pi} \int_t^{t+2\pi/\omega} \exp[i(E_0/\omega) \sin(\omega u)] du$$

The above equation tells us that the tunneling parameter is reduced effectively by the factor $J_0(E_0/\omega)$, and even vanishes in the case that E_0/ω coincides with a zero of the Bessel function. This is the CDT[6].

An infinite dimensional analogue of the model (1) is given by the Hamiltonian

$$\begin{aligned} H_2(t) &= E(t) \sum_{n=-\infty}^{\infty} n |n\rangle \langle n| \\ &+ \Delta \sum_{n=-\infty}^{\infty} (|n\rangle \langle n+1| + |n+1\rangle \langle n|). \end{aligned} \quad (5)$$

This is a model Hamiltonian for an electron in an infinite one-dimensional chain under a time-dependent electric field, where $|n\rangle$ represents the Wannier state at site n . Paradoxically, the Schrödinger equation $id/dt|\varphi(t)\rangle = H_2(t)|\varphi(t)\rangle$ is solved analytically for arbitrary functional forms of $E(t)$. We show explicit time-evolution operator with a Lie algebra. We define $T_0 = \sum_{n=-\infty}^{\infty} n |n\rangle \langle n|$, $T_+ = \sum_{n=-\infty}^{\infty} |n+1\rangle \langle n|$, $T_- = \sum_{n=-\infty}^{\infty} |n\rangle \langle n+1|$ to get $H_2(t) = E(t)T_0 + \Delta(T_+ + T_-)$. These operators satisfies the relations:

$$[T_0, T_{\pm}] = \pm T_{\pm}, \quad [T_+, T_-] = 0. \quad (6)$$

The solution of the Schrödinger equation is written as $|\varphi(t)\rangle = U(t)|\varphi(0)\rangle$, where $U(t) = \exp_+(-i \int_0^t H_2(s) ds)$. By Feynman's disentangling theorem, $U(t)$ is written in the form,

$$U(t) = e^{-iA(t)T_0} \exp_+ \left[-i\Delta \int_0^t (\tilde{T}_+(u) + \tilde{T}_-(u)) du \right],$$

where $A(t) \equiv \int_0^t E(u) du$, and $\tilde{T}_{\pm}(u) \equiv e^{iA(u)T_0} T_{\pm} e^{-iA(u)T_0} = e^{\pm iA(u)} T_{\pm}$. Since T_+ and T_- are commutable, $U(t)$ is rewritten as

$$U(t) = \exp[-iA(t)T_0] \exp[-iB(t)] \quad (7)$$

in which $B(t) = \Delta \{R(t)T_+ + R(t)^*T_-\}$ with $R(t) = \int_0^t \exp[iA(u)] du$. Since $B(t)$ has the translational symmetry, its eigenstates are given by the plane waves $|k\rangle = \sum_n e^{ikn} |n\rangle$ with the time-dependent eigenvalue

$\epsilon_k(t) = \Delta \{R(t)e^{-ik} + R^*(t)e^{ik}\}$. Then the matrix element for the transition $|n\rangle \rightarrow |m\rangle$ is calculated by using the closure relation as

$$\begin{aligned} \langle m|U(t)|n\rangle &= \exp \left[-iA(t)m + i \left(\chi + \frac{\pi}{2} \right) (m-n) \right] \\ &\times J_{m-n}(2\Delta|R(t)|), \end{aligned} \quad (8)$$

where $\chi = \arg R(t)$ and $J_n(x)$ is the n th order Bessel function. For a specific choice $E(t) = E_0 \cos(\omega t)$, and at each period of the oscillation $\tau = 2\pi l/\omega$ ($l = 0, 1, 2, \dots$), we find $A(\tau) = 0$ and $R(\tau) = \tau J_0(E_0/\omega)$, and the transition probability is given by

$$|\langle m|U(\tau)|n\rangle|^2 = J_{m-n}^2(2\tau\Delta|J_0(E_0/\omega)|). \quad (9)$$

This should be compared with the value $J_{m-n}^2(2\tau\Delta)$ which corresponds to the case without external field. Eq.(9) indicates that the oscillating external field generally reduces the effective transfer by the factor $J_0(E_0/\omega)$. Especially, if E_0/ω coincides with a zero of $J_0(x)$, the probability to find the electron at site $m(\neq n)$ oscillates temporally and becomes zero, while that to find it at the original site n becomes unity at each period $2\pi/\omega$. This is the dynamic localization (DL)[7].

It is clear that the integrability of the Schrödinger equation for (5) rests upon the commutativity of T_+ and T_- . On the other hand, for the two-state model (1), we can define the analogous operators, $S_0 = \frac{1}{2}(|1\rangle\langle 1| - |2\rangle\langle 2|)$, $S_+ = |1\rangle\langle 2|$, and $S_- = |2\rangle\langle 1|$, to get $H_1(t) = E(t)S_0 + \gamma(S_+ + S_-)$. These operators, however, satisfy a true SU(2) Lie algebra:

$$[S_0, S_{\pm}] = \pm S_{\pm}, \quad [S_+, S_-] = 2S_0. \quad (10)$$

These are uncommutable relations and the time-evolution operator cannot be decomposed in general.

Let us discuss the relations between the dynamics of CDT and DL. We consider the following bosonic representation for (1) with Schwinger bosons:

$$H_3(t) = \frac{E(t)}{2} (b_1^\dagger b_1 - b_2^\dagger b_2) + \gamma (b_1^\dagger b_2 + b_2^\dagger b_1), \quad (11)$$

where b_i satisfies the commutation relation of independent bosons, $[b_i, b_j^\dagger] = \delta_{i,j}$. The Heisenberg equation for b_1^\dagger and b_2^\dagger is given by

$$\begin{aligned} i \frac{d}{dt} b_1^\dagger(t) &= -\frac{E(t)}{2} b_1^\dagger(t) - \gamma b_2^\dagger(t), \\ i \frac{d}{dt} b_2^\dagger(t) &= \frac{E(t)}{2} b_2^\dagger(t) - \gamma b_1^\dagger(t), \end{aligned} \quad (12)$$

which is equivalent to Eq.(2) by the replacement $b_i^\dagger(t)$ by $a_i(t)$. The solution of Eq.(12) with the initial conditions, $b_1^\dagger(0) = b_1^\dagger$, and $b_2^\dagger(0) = b_2^\dagger$ is generally written as

$$\begin{aligned} \begin{pmatrix} b_1^\dagger(t) \\ b_2^\dagger(t) \end{pmatrix} &= U_b \begin{pmatrix} b_1^\dagger \\ b_2^\dagger \end{pmatrix} \\ U_b &= \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \end{aligned} \quad (13)$$

where α and β are time-dependent complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$. The point is that, if the two-state dynamics described by Eq. (2) is solved somehow, it can be mapped onto the solution for Eq. (12) and we obtain a class of solutions for state vectors in higher dimensional representation spaces of SU(2). Recently, Pokrovsky and Sinitsyn[12] utilized the same argument to derive a class of exact formulas describing the nonadiabatic transitions for a model of multiple level crossings.

Let us define the basis states designated by the boson numbers,

$$|\Psi\rangle = |p, q\rangle = \frac{1}{\sqrt{p!q!}} b_1^{\dagger p} b_2^{\dagger q} |\text{vac}\rangle, \quad (14)$$

where $|\text{vac}\rangle$ is the vacuum state of the bosons. The total boson number is a constant of motion. We fix $p+q = 2N$, and define the *site* index n by $n \equiv (p-q)/2$. The basis states are classified as $|n\rangle = |N+n, N-n\rangle, (n = -N, -N+1, \dots, 0, 1, \dots, N)$. The nonzero off-diagonal matrix elements are then given by $\langle n+1|H_3|n\rangle = \gamma\sqrt{(N+n+1)(N-n)}$. If we set $\gamma = \Delta/N$, we have a $2N+1$ -dimensional linear chain model as a representation of the SU(2) Hamiltonian,

$$H_3(t) = E(t) \sum_{n=-N}^N n|n\rangle\langle n| + \Delta \sum_{n=-N}^{N-1} f_n (|n+1\rangle\langle n| + |n\rangle\langle n+1|), \quad (15)$$

in which $f_n = \sqrt{(1 + \frac{n+1}{N})(1 - \frac{n}{N})}$. Also in the sector $p+q = 2N-1$, an analogous expression is obtained. Specifically, for $p+q = 1$, the two-state model $H_1(t)$ is recovered. An important observation here is that, in the limit $N \rightarrow \infty$ with fixed n , the tight-binding model with an infinite chain $H_2(t)$ is also recovered since $f_n \rightarrow 1$. Thus the CDT dynamics in U_b can be connected to the DL dynamics in the wave functions for the Hamiltonian (15).

We now study the time-evolution operator for the wave function, $V(t)$, which satisfies, $|\Psi(t)\rangle = V(t)|\Psi\rangle$. Once explicit matrix elements in U_b are obtained, one obtains a class of time-evolutions for the driven system (15). The wave function $|\Psi(t)\rangle$ is given with U_b as

$$|\Psi(t)\rangle = \frac{1}{\sqrt{p!q!}} (\alpha^* b_1^\dagger - \beta b_2^\dagger)^p (\beta^* b_1^\dagger + \alpha b_2^\dagger)^q |\text{vac}\rangle. \quad (16)$$

By expanding the right hand side, and rearranging the terms proportional to $b_1^{\dagger N+m} b_2^{\dagger N-m}$, we find the transi-

tion amplitude for $|n\rangle \rightarrow |m\rangle$,

$$\begin{aligned} \langle m|V(t)|n\rangle &= \sqrt{\frac{(N+m)!(N-m)!}{(N+n)!(N-n)!}} \alpha^{*n+m} \beta^{*m-n} \\ &\times \sum_{r=r_m}^{r_M} \binom{N+n}{r} \binom{N-n}{N-n-r} \\ &\times |\alpha|^{2(N-m-r)} (-|\beta|^2)^r, \end{aligned} \quad (17)$$

where the summation over r runs from $r_m = \max\{0, n-m\}$ to $r_M = \min\{N+n, N-m\}$. This is rewritten as,

$$\begin{aligned} \langle m|V(t)|n\rangle &= \sqrt{\frac{(N+m)!(N-m)!}{(N+n)!(N-n)!}} \alpha^{*m+n} \beta^{*m-n} P_{N-m}^{m-n, m+n}(x), \end{aligned} \quad (18)$$

where $x = 2|\alpha|^2 - 1$, and $P_{N-m}^{m-n, m+n}(x)$ is Jacobi's polynomial[13] defined as,

$$P_n^{(a,b)}(x) = \frac{1}{2^n} \sum_{r=0}^n \binom{n+a}{n-r} \binom{n+b}{r} (x-1)^r (x+1)^{n-r}.$$

This expression of $V(t)$ is valid in the region $m-n \geq 0, m+n \geq 0$. In other regions, $\langle m|V(t)|n\rangle$ is given by the replacement; $m \rightarrow -n, n \rightarrow -m, \alpha^* \rightarrow \alpha$ for $m-n \geq 0, m+n \leq 0, m \rightarrow n, n \rightarrow m, \beta^* \rightarrow -\beta$ for $m-n \leq 0, m+n \geq 0$, and $m \rightarrow -m, n \rightarrow -n, \alpha^* \rightarrow \alpha, \beta^* \rightarrow \beta$ for $m-n \leq 0, m+n \leq 0$.

Now set $E(t)$ to a sinusoidal modulation, $E(t) = E_0 \cos(\omega t)$ with $\gamma = \Delta/N$. The condition for the rapid modulation limit $\omega \gg \gamma$ is satisfied for Eq. (12) in the limit $N \gg 1$, and it is solved just the same way as the corresponding equation for the c-numbers (2). We obtain

$$\begin{aligned} \alpha(\tau) &= \exp \left[i \frac{E_0}{2\omega} \sin \omega \tau \right] \cos \left(\frac{\Delta}{N} J_0(E_0/\omega) \tau \right), \\ \beta(\tau) &= i \exp \left[i \frac{E_0}{2\omega} \sin \omega \tau \right] \sin \left(\frac{\Delta}{N} J_0(E_0/\omega) \tau \right). \end{aligned} \quad (19)$$

Note that the time τ is coarse-grained by the unit $2\pi/\omega$. The following formula is easily proved by using Stirling's formula[14],

$$\lim_{N \rightarrow \infty} N^{-a} P_N^{(a,b)} \left(1 - \frac{z^2}{2N^2} \right) = \left(\frac{z}{2} \right)^{-a} J_a(|z|). \quad (20)$$

Then, inserting Eq.(19) into Eq.(18), and noting that $x = \cos(2\Delta/N J_0(E_0/\omega)t)$, we get, in the limit $N \rightarrow \infty$,

$$\begin{aligned} \langle m|V(\tau)|n\rangle &= \exp \left[-i \frac{E_0 m}{\omega} \sin \omega \tau + i \frac{\pi}{2} (m-n) \right] \\ &\times J_{m-n}(2\Delta\tau |J_0(E_0/\omega)|). \end{aligned} \quad (21)$$

This is exactly the same as the formula (8) including the phase factor. Especially, when E_0/ω coincides with a zero

of $J_0(E_0/\omega)$, the CDT occurs in U_b , while the DL occurs in $V(t)$. Thus it is shown that the DL is an infinitely large dimensional representation of a generalized version of the CDT.

One of the special cases of a class of the Hamiltonian (1) that allows the exact solution is the Landau-Zener model[15, 16] $E(t) = vt$. The solution is written in terms of Weber functions, and the transition probability from one branch to another according to the temporal evolution from $t = -\infty$ to $t = \infty$ is given by the celebrated Landau-Zener formula[15, 16]. One of the present authors[17] pointed out that CDT can be regarded as a result of destructive interference between the transition paths for repeated Landau-Zener level crossings. The above result suggests the possibility to extend this view to the DL. In the case $E(t) = vt$ ($v > 0$), the transition matrix elements without adiabatic phases are given by

$$\alpha = \sqrt{P}, \quad \beta = -\sqrt{1-P}e^{i\phi} \quad (22)$$

where $P = \exp[-2\pi\delta]$ is the Landau-Zener nonadiabatic transition probability with $\delta = \Delta^2/(N^2v)$, and ϕ is the Stokes phase given by $\phi = \pi/4 + \arg \Gamma(1-i\delta) + \delta(\ln \delta - 1)$ in which $\Gamma(z)$ is the Γ function. The transfer matrix for the two-state Landau-Zener model can be mapped onto the $2N + 1$ -site representation S as before. Noting that, in the limit $N \gg 1$, $\alpha^2 \simeq 1 - 2\pi\delta$ and $\beta \simeq -\sqrt{2\pi\delta}e^{i\pi/4}$, we find for the matrix element $\langle m|S|n \rangle$ at the crossing

$$\langle m|S|n \rangle = \exp\left[-i\frac{\pi}{4}(m-n)\right] J_{m-n}\left(2\sqrt{2\pi\Delta/v}\right). \quad (23)$$

This formula agrees with the exact formula obtained from Eq.(8), as it should.

For a repeated crossings of the two-state model driven by $E(t) = E_0 \cos(\omega t)$, and in the case that E_0 is much larger than γ and ω , we can approximately decompose the whole process into sudden transitions at level-crossings and the free propagation between them[17]. The velocity of energy change v is given by the value estimated at the crossings, $v = E_0\omega$. This is also mapped

onto the $2N + 1$ -dimensional representation. Thus, for a double crossing within a period of the oscillation, say at $t_1 = \pi/2\omega$ and $t_2 = 3\pi/2\omega$, we have the transition amplitude $\langle m|T|n \rangle$ from $|n \rangle$ to $|m \rangle$ in the limit $N \rightarrow \infty$ as a sum of all contribution from the intermediate states,

$$\langle m|T|n \rangle = \sum_{l=-\infty}^{\infty} \langle m|S|l \rangle e^{-i\Omega l} \langle l|S^T|n \rangle, \quad (24)$$

where S^T is the transpose of S , and $\Omega \equiv \int_{t_1}^{t_2} dt E_0 \cos(\omega t) = 2E_0/\omega$. The summation is carried out exactly by using Graf's formula[18],

$$\begin{aligned} & \sum_{m=-\infty}^{\infty} J_{\nu+m}(z) J_m(\zeta) e^{im\theta} \\ &= J_{\nu} \left(\sqrt{z^2 + \zeta^2 - 2z\zeta \cos \theta} \right) \left\{ \frac{z - \zeta e^{-i\theta}}{z - \zeta e^{i\theta}} \right\}^{\nu/2}, \end{aligned}$$

valid for real numbers z and ζ . The transition probability is thus obtained as

$$|\langle m|T|n \rangle|^2 = J_{m-n}^2 \left[2\Delta \sqrt{2\omega/\pi E_0} \sin \left(\frac{E_0}{\omega} + \frac{\pi}{4} \right) \frac{2\pi}{\omega} \right]. \quad (25)$$

If one notices the asymptotic formula $J_0(x) \simeq \sqrt{2/\pi x} \sin(x + (\pi/4))$ for $x \gg 1$, it can be seen that the formula (9) agrees at $t = 2\pi/\omega$ with the above one in the limit $E_0/\omega \gg 1$. The phase factor $\pi/4$ is nothing but the Stokes phase at the level-crossing in the diabatic limit. Thus it is revealed that, in the level of the two-state model, the CDT is a result of interference between the two intermediate transition paths, while in its infinitely large dimensional representation, the DL is a result of interference between infinite number of intermediate transition paths.

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